

# Some Invariant String Cosmological Models in Cylindrically Symmetric Space-time

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## Abstract

In this paper we derive some new invariant solutions of Einstein-Maxwell's field equations for string fluid as source of matter in cylindrically symmetric space-time with Variable Magnetic Permeability. We also discuss the physical and geometrical properties of the models derived in the paper. The solutions, at least one of them, are interesting physically as they can explain the accelerating as well as singularity free Universe.

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## 1 Introduction

One of the important consequence of general theory of relativity is to know the large scale structure of the Universe. The study of cosmology is devoted to the Universe as a whole. Initial study was based on FRW type which describes the space which is homogeneous and isotropic. We know homogeneity and isotropy are symmetries of space. Though the present observations indicates that the Universe is almost perfectly homogeneous and isotropic in large scales, however, the symmetries of space in very early Universe could be very different. Today's observations could not able to provide information about symmetries of space near initial singularity. Hence, it is very justified to consider inhomogeneous and anisotropic models of the Universe. Cylindrically-symmetric space-time is more general than the homogeneous and isotropic space-time and plays an important role in the study of the universe when the anisotropy and inhomogeneity are taking into consideration. Barrow and Kunze [1, 2] investigated flat and open homogeneous universe, considering string fluid as source of matter. In the recent past, Pradhan et al [3] and Yadav et al [4] have studied inhomogeneous cosmological models with cloud of strings; the latter were invoked to palliate the problems associated with cylindrically symmetric space-time.

Due to non-linearity, general theory of relativity is a very difficult theory. Therefore, to describe the models of the Universe, different group of symmetries are used in literature. In this study, we apply the so-called symmetry analysis method. The main advantage of this method is that it can be successfully applied to nonlinear differential equations. The similarity solutions are attractive because they result in the reduction of the independent variables of the problem. In our case, the problem under investigation is the system of second order nonlinear PDEs and similarity solution will transform this system of nonlinear PDEs into a system of ODEs.

The groups of continuous transformations that leave a given family of equations invariant is known as symmetry groups (isovector fields) [5, 6, 7, 8, 9]. In a pioneering work, Ovsiannikov [10] mentioned that the usual Lie infinitesimal invariance approach could as well be employed in order to construct symmetry groups [11, 12, 13].

In recent years, there has been considerable interest in string cosmology. Cosmic string plays an important role in the study of the early universe. String cosmology is a relatively new field that tries to apply equation of string theory to solve the questions of early cosmology, a related area of study in brain cosmology. It was argued by Melvin [14] that the presence of magnetic field is not as unrealistic because for a large part of the history of evolution matter was highly ionized and matter and field were smoothly coupled. Latter during the expansion of the Universe, the ions combined to form neutral matter. Therefore, inclusion of the magnetic field with Variable Magnetic Permeability is justified for cosmological modeling of the Universe.

In this paper, we attempted to find a new class of exact (similarity) solutions for string cosmological models in cylindrically symmetric inhomogeneous universe with electro magnetic perfect fluid distribution with variable magnetic permeability in general relativity.

We organize the paper as follows: String cosmological model in cylindrically symmetric inhomogeneous universe with electro-magnetic perfect fluid distribution of variable magnetic permeability in general relativity, is introduced in section 2. In section 3, symmetry analysis and isovector fields for Einstein field equations are obtained. In section 4, we find a new class of exact (similarity) solutions for Einstein field equations. Section 5 is devoted to the study of some physical and geometrical properties of the obtained model. The paper ends with a short discussion.

## 2 The metric and field equations

We consider the metric in the form

$$ds^2 = A^2 (dx^2 - dt^2) + B^2 dy^2 + C^2 dz^2, \quad (1)$$

where  $A$ ,  $B$  and  $C$  are functions of  $x$  and  $t$ . The energy-momentum tensor for the string with electro-magnetic field has the form

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j + E_{ij}, \quad (2)$$

where  $u_i$  and  $x_i$  satisfy the conditions

$$u^i u_i = -x^i x_i = -1, \quad (3)$$

and

$$u^i x_i = 0. \quad (4)$$

Here  $\rho$  being the rest energy density of the system of strings,  $\lambda$  the tension density of the strings,  $x^i$  is a unit space-like vector representing the direction of strings so that  $x^1 = x^2 = x^4 = 0$  and  $x^3 \neq 0$ , and  $u^i$  is the four velocity vector.  $E_{ij}$  is the electro-magnetic field given by Lichnerowicz [15]:

$$E_{ij} = \bar{\mu} [h_i h^l (u_j u_l + \frac{1}{2} g_{jl}) - h_i h_j], \quad (5)$$

where  $\bar{\mu}$  is the magnetic permeability and  $h_i$  the magnetic flux vector defined by:

$$h_i = \frac{1}{\bar{\mu}} {}^*F_{ji} u^j, \quad (6)$$

where the dual electro-magnetic field tensor  ${}^*F_{ij}$  is defined by Synge [16]

$${}^*F_{ij} = \frac{\sqrt{-g}}{2} \epsilon_{ijkl} F^{kl}. \quad (7)$$

Here  $F_{ij}$  is the electro-magnetic field tensor and  $\epsilon_{ijkl}$  is a Levi-Civita tensor density. In the present scenario, the co-moving coordinates are taken as

$$u^i = \left(0, 0, 0, \frac{1}{A}\right). \quad (8)$$

We choose the direction of string parallel to  $x$ -axis so that

$$x^i = \left(\frac{1}{A}, 0, 0, 0\right). \quad (9)$$

If we consider the current flow along  $z$ -axis, then  $F_{12}$  is only non-vanishing component of  $F_{ij}$ . Then the Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \quad (10)$$

and

$$\left[\frac{1}{\mu}F^{ij}\right]_{;j} = J^i \quad (11)$$

require that  $F_{12}$  be function of  $x$  alone [17]. We assume that the magnetic permeability as a function of both  $x$  and  $t$ . Here the semicolon represents a covariant differentiation.

The Einstein's field equation

$$R_{ij} - \frac{1}{2} g_{ij} R = -\chi T_{ij}, \quad (12)$$

for the line-element (1) lead to the following system of equations:

$$\frac{C_{xt}}{C} + \frac{B_{xt}}{B} - \frac{A_t C_x + A_x C_t}{AC} - \frac{A_t B_x + A_x B_t}{AB} = 0, \quad (13)$$

$$\frac{C_{xx} - C_{tt}}{2C} + \frac{B_{xx} - B_{tt}}{2B} + \frac{A_{xx} - A_{tt}}{A} + \frac{A_t^2 - A_x^2}{A^2} = 0, \quad (14)$$

$$\chi A^2 \lambda = \frac{C_{xx}}{C} + \frac{B_{tt}}{B} + \frac{A_{xx} - A_{tt}}{A} + \frac{B_t C_t - B_x C_x}{BC} \quad (15)$$

$$- \frac{A_t C_t + A_x C_x}{AC} - \frac{A_x B_x + A_t B_t}{AB} + \frac{A_t^2 - A_x^2}{A^2},$$

$$\chi A^2 \rho = \frac{C_{tt} - 2C_{xx}}{C} - \frac{B_{tt}}{B} + \frac{A_{tt} - A_{xx}}{A} + \frac{B_t C_t - B_x C_x}{BC} \quad (16)$$

$$+ \frac{A_t C_t + A_x C_x}{AC} + \frac{A_x B_x + A_t B_t}{AB} + \frac{A_t^2 - A_x^2}{A^2},$$

$$\frac{\chi F_{12}^2}{2\mu B^2} = \frac{C_{xx} - C_{tt}}{C} + \frac{A_{xx} - A_{tt}}{A} + \frac{A_t^2 - A_x^2}{A^2}, \quad (17)$$

The velocity field  $u^i$  is ir-rotational. The scalar expansion  $\Theta$ , shear scalar  $\sigma^2$ , acceleration vector  $\dot{u}_i$  and Proper volume  $V$  are respectively found to have the following expressions [18, 19]:

$$\Theta = u_{;i}^i = \frac{1}{A} \left( \frac{C_t}{C} + \frac{B_t}{B} + \frac{A_t}{A} \right), \quad (18)$$

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{\Theta^2}{3} - \frac{1}{A^2} \left( \frac{B_t C_t}{BC} + \frac{A_t C_t}{AC} + \frac{A_t B_t}{AB} \right), \quad (19)$$

$$\dot{u}_i = u_{i;j} u^j = \left( \frac{A_x}{A}, 0, 0, 0 \right), \quad (20)$$

$$V = \sqrt{-g} = A^2 B C, \quad (21)$$

where  $g$  is the determinant of the metric (1). The shear tensor is

$$\sigma_{ij} = u_{(i;j)} + \dot{u}_{(i} u_{j)} - \frac{1}{3} \Theta (g_{ij} + u_i u_j). \quad (22)$$

and the non-vanishing components of the  $\sigma_i^j$  are

$$\left\{ \begin{array}{l} \sigma_1^1 = \frac{1}{3A} \left( \frac{2A_t}{A} - \frac{B_t}{B} - \frac{C_t}{C} \right), \\ \sigma_2^2 = \frac{1}{3A} \left( \frac{2B_t}{B} - \frac{C_t}{C} + \frac{A_t}{A} \right), \\ \sigma_3^3 = \frac{1}{3A} \left( \frac{2C_t}{C} - \frac{B_t}{B} + \frac{A_t}{A} \right), \\ \sigma_4^4 = 0. \end{array} \right. \quad (23)$$

Using the field equations and the relations (18) and (19) one obtains the Raychaudhuri's equation is

$$\frac{\partial \Theta}{\partial t} = \dot{u}_{;i}^i - \frac{\Theta^2}{3} - 2\sigma^2 - \frac{\rho_p}{2}, \quad (24)$$

where

$$\rho_p = 2 R_{ij} u^i u^j. \quad (25)$$

The Einstein field equations (13)-(17) constitute a system of five highly non-linear differential equations with six unknowns variables,  $A$ ,  $B$ ,  $C$ ,  $\lambda$ ,  $\rho$  and  $\frac{F_{\mu}^2}{\mu}$ . Therefore, one physically reasonable conditions amongst these parameters are required to obtain explicit solutions of the field equations. Let us assume that the expansion scalar  $\Theta$  in the model (1) is proportional to the eigenvalue  $\sigma_1^1$  of the shear tensor  $\sigma_j^k$ . Then from (18) and (23), we get

$$\frac{2A_t}{A} - \frac{B_t}{B} - \frac{C_t}{C} = 3\gamma \left( \frac{A_t}{A} + \frac{B_t}{B} + \frac{C_t}{C} \right), \quad (26)$$

where  $\gamma$  is a constant of proportionality. The above equation can be written in the form

$$\frac{A_t}{A} = n \left( \frac{B_t}{B} + \frac{C_t}{C} \right). \quad (27)$$

where  $n = \frac{1+3\gamma}{2-3\gamma}$ . If we integrate the above equation with respect to  $t$ , we can get the following relation

$$A(x, t) = f(x) \left( B(x, t) C(x, t) \right)^n, \quad (28)$$

where  $f(x)$  is a constant of integration which is an arbitrary function of  $x$ . If we substitute the metric function  $A$  from (20) in the Einstein field equations, the equations (13)-(14) transform to the nonlinear partial differential equations of the coefficients  $B$  and  $C$  only, as the following new form:

$$E_1 = \frac{B_{xt}}{B} + \frac{C_{xt}}{C} - 2n \left( \frac{B_x B_t}{B^2} + \frac{B_x C_t + B_t C_x}{BC} + \frac{C_x C_t}{C^2} \right) - \frac{f'}{f} \left( \frac{B_t}{B} + \frac{C_t}{C} \right) = 0, \quad (29)$$

$$E_2 = \left( n + \frac{1}{2} \right) \left[ \frac{B_{xx} - B_{tt}}{B} + \frac{C_{xx} - C_{tt}}{C} \right] + n \left( \frac{B_t^2 - B_x^2}{B^2} + \frac{C_t^2 - C_x^2}{C^2} \right) + \frac{f''}{f} - \frac{f'^2}{f^2} = 0, \quad (30)$$

where the prime indicates derivative with respect to the coordinate  $x$ .

### 3 Symmetry analysis method

In order to obtain an exact solutions of the system of nonlinear partial differential equations (29)-(30), we will use the symmetry analysis method. For this we write

$$\begin{cases} x_i^* = x_i + \epsilon \xi_i(x_j, u_\beta) + \mathbf{o}(\epsilon^2), \\ u_\alpha^* = u_\alpha + \epsilon \eta_\alpha(x_j, u_\beta) + \mathbf{o}(\epsilon^2), \end{cases} \quad i, j, \alpha, \beta = 1, 2, \quad (31)$$

as the infinitesimal Lie point transformations. We have assumed that the system (29)-(30) is invariant under the transformations given in Eq. (31). The corresponding infinitesimal generator of Lie groups (symmetries) is given by

$$X = \sum_{i=1}^2 \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^2 \eta_\alpha \frac{\partial}{\partial u_\alpha}, \quad (32)$$

where  $x_1 = x$ ,  $x_2 = t$ ,  $u_1 = B$  and  $u_2 = C$ . The coefficients  $\xi_1$ ,  $\xi_2$ ,  $\eta_1$  and  $\eta_2$  are the functions of  $x$ ,  $t$ ,  $B$  and  $C$ . These coefficients are the components of infinitesimals symmetries corresponding to  $x$ ,  $t$ ,  $B$  and  $C$  respectively, to be determined from the invariance conditions:

$$\text{Pr}^{(2)} X \left( E_m \right) |_{E_m=0} = 0, \quad (33)$$

where  $E_m = 0$ ,  $m = 1, 2$  are the system (29)-(30) under study and  $\text{Pr}^{(2)}$  is the second prolongation of the symmetries  $X$ . Since our equations (29)-(30) are at most of order two, therefore, we need second order prolongation of the infinitesimal generator in Eq. (33). It is worth noting that, the 2-th order prolongation is given by:

$$\text{Pr}^{(2)} X = \sum_{i=1}^2 \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^2 \eta_\alpha \frac{\partial}{\partial u_\alpha} + \sum_{i=1}^2 \sum_{\alpha=1}^2 \eta_{\alpha i} \frac{\partial}{\partial u_{\alpha,i}} + \sum_{j=1}^2 \sum_{i=1}^2 \sum_{\alpha=1}^2 \eta_{\alpha i j} \frac{\partial}{\partial u_{\alpha,i j}}, \quad (34)$$

where

$$\eta_{\alpha i} = D_i(\eta_\alpha) - \sum_{j=1}^2 u_{\alpha,j} D_i(\xi_j), \quad \eta_{\alpha i j} = D_j(\eta_{\alpha i}) - \sum_{k=1}^2 u_{\alpha,k i} D_j(\xi_k). \quad (35)$$

The operator  $D_i$  is called the *total derivative* (*Hach operator*) and taken the following form:

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^2 u_{\alpha,i} \frac{\partial}{\partial u_\alpha} + \sum_{j=1}^2 \sum_{\alpha=1}^2 u_{\alpha,j i} \frac{\partial}{\partial u_{\alpha,j}}, \quad (36)$$

where  $u_{\alpha,i} = \frac{\partial u_\alpha}{\partial x_i}$  and  $u_{\alpha,i j} = \frac{\partial^2 u_\alpha}{\partial x_j \partial x_i}$ .

Expanding the system of Eqs. (33) along with the original system of Eqs. (29)-(30) to eliminate  $B_{xx}$  and  $B_{xt}$  while we set the coefficients involving  $C_x$ ,  $C_t$ ,  $C_{xx}$ ,  $C_{xt}$ ,  $C_{tt}$ ,  $B_x$ ,  $B_t$ ,  $B_{tt}$  and various products to zero give rise the essential set of over-determined equations. Solving the set of these determining equations, the components of symmetries takes the following form:

$$\xi_1 = c_1 x + c_2, \quad \xi_2 = c_1 t + c_3, \quad \eta_1 = c_4 B, \quad \eta_2 = c_5 C, \quad (37)$$

such that the function  $A(t)$  must be equal:

$$\begin{cases} f(x) = c_6 \exp[c_7 x], & \text{if } c_1 = 0, \\ f(x) = c_8 (c_1 x + c_2)^{c_9}, & \text{if } c_1 \neq 0, \end{cases} \quad (38)$$

where  $c_i$ ,  $i = 1, 2, \dots, 9$  are an arbitrary constants.

## 4 Similarity solutions

The characteristic equations corresponding to the symmetries (37) are given by:

$$\frac{dx}{c_1 x + c_2} = \frac{dt}{c_1 t + c_3} = \frac{dB}{c_4 B} = \frac{dC}{c_5 C}. \quad (39)$$

By solving the above system, we have the following two cases:

**Case (1):** When  $c_1 = 0$ , the similarity variable and similarity functions can be written as the following:

$$\xi = a x + b t, \quad B(x, t) = \Psi(\xi) \exp[c x], \quad C(x, t) = \Phi(\xi) \exp[d x], \quad (40)$$

where  $a = c_3$ ,  $b = -c_2$ ,  $c = \frac{c_4}{c_2}$  and  $d = \frac{c_5}{c_2}$  are an arbitrary constants. Substituting the transformations (40) in the field Eqs. (20)-(21) lead to the following system of ordinary differential equations:

$$\frac{a \Psi'' + [c_7 - c + 2n(c+d)] \Psi'}{\Psi} + \frac{a \Phi'' + [c_7 - d + 2n(c+d)] \Phi'}{\Phi} = 2an \left( \frac{\Psi'}{\Psi} + \frac{\Phi'}{\Phi} \right)^2, \quad (41)$$

$$(b^2 - a^2) \left[ (2n+1) \left( \frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi} \right) - 2n \left( \frac{\Psi'^2}{\Psi^2} + \frac{\Phi'^2}{\Phi^2} \right) \right] - 2a \left( \frac{c \Psi'}{\Psi} + \frac{d \Phi'}{\Phi} \right) = c^2 + d^2, \quad (42)$$

The equations (41) and (42) are non-linear ordinary differential equations which is very difficult to solve. However, it is worth noting that, this equation is easy to solve when  $b = a$ . In this case, the equation (42) takes the form:

$$\frac{c \Psi'}{\Psi} + \frac{d \Phi'}{\Phi} = -\frac{c^2 + d^2}{2a}. \quad (43)$$

By integration the above equation, we can get the following:

$$\Phi(\xi) = q_1 \Psi^{\alpha_1}(\xi) \exp[\alpha_2 \xi], \quad (44)$$

where  $\alpha_1 = -\frac{c}{d}$  and  $\alpha_2 = -\left(\frac{c^2 + d^2}{2ad}\right)$  while  $q_1$  is an arbitrary constant of integration. Substitute (44) in (41), we have the following ordinary differential equation of the function  $\Psi$  only as follows:

$$\frac{\Psi''}{\Psi} = (\alpha_3 - 1) \left(\frac{\Psi'^2}{\Psi^2}\right) + \alpha_4 \left(\frac{\Psi'}{\Psi}\right) + \alpha_5, \quad (45)$$

where

$$\begin{cases} \alpha_3 = -\frac{1 + \alpha_1^2 - 2n(1 + \alpha_1)^2}{1 + \alpha_1}, \\ \alpha_4 = -\frac{2\alpha_2 \left[ a_7(1 + \alpha_1) + d\alpha_1 \left( 1 + \alpha_1^2 - 2n(1 + \alpha_1)^2 \right) \right]}{d(1 + \alpha_1)(1 + \alpha_1^2)}, \\ \alpha_5 = -\frac{\alpha_2^2 \left[ 2a_7 + d \left( \alpha_1^2 - 1 - 2n(\alpha_1^2 + 2\alpha_1 - 1) \right) \right]}{d(1 + \alpha_1)(1 + \alpha_1^2)}. \end{cases} \quad (46)$$

If we use the transformation

$$\Psi(\xi) = q_2 \exp \left[ \int \Omega(\xi) d\xi \right] \quad (47)$$

the equation (45) becomes:

$$\Omega' = \alpha_3 \Omega^2 + \alpha_4 \Omega + \alpha_5, \quad (48)$$

where  $q_2$  is constant while  $\Omega(\xi)$  is a new function of  $\xi$ . For solving the above ordinary differential equation, we must take the following cases:

**Case (1.1):** When  $\alpha_3 \neq 0$ ,  $\alpha_4 \neq 0$  and  $\alpha_5 \neq 0$ , there exists three cases as the following:

**Case (1.1.1):** When  $4\alpha_3\alpha_5 - \alpha_4^2 = \frac{4K_1^2}{a^2}$ , the general solution of the equation (48) is:

$$\Omega(\xi) = -\frac{\alpha_4}{2\alpha_3} - \frac{K_1}{a\alpha_3} \tan \left[ \frac{K_1\xi}{a} + \xi_0 \right], \quad (49)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.1.2):** When  $\alpha_4^2 - 4\alpha_3\alpha_5 = \frac{4K_2^2}{a^2}$ , the general solution of the equation (48) is:

$$\Omega(\xi) = -\frac{\alpha_4}{2\alpha_3} + \frac{K_2}{a\alpha_3} \tanh \left[ \frac{K_2\xi}{a} + \xi_0 \right], \quad (50)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.1.3):** When  $\alpha_4^2 = 4\alpha_3\alpha_5$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \frac{\alpha_4\xi - 2}{2\alpha_3\xi}. \quad (51)$$

**Case (1.2):** When  $\alpha_3 \neq 0$ ,  $\alpha_4 \neq 0$  and  $\alpha_5 = 0$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \frac{\alpha_4}{\exp \left[ \alpha_3 - \alpha_4(\xi + \xi_0) \right]}, \quad (52)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.3):** When  $\alpha_3 \neq 0$ ,  $\alpha_4 = 0$  and  $\alpha_5 \neq 0$ , then there exists the following cases:

**Case (1.3.1):** When  $\alpha_3 = \pm K_4^2$  and  $\alpha_5 = \pm K_3^2$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \pm \frac{K_3}{K_4} \tan \left[ K_4 K_3 (\xi + \xi_0) \right], \quad (53)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.3.2):** When  $\alpha_3 = \mp K_6^2$  and  $\alpha_5 = \pm K_5^2$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \pm \frac{K_5}{K_6} \tanh \left[ K_5 K_6 (\xi + \xi_0) \right], \quad (54)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Remark (1):** The two cases (1.3.1) and (1.3.2) are the spacial cases from the two cases (1.1.1) and (1.1.2), respectively.

**Case (1.4):** When  $\alpha_3 = 0$ ,  $\alpha_4 \neq 0$  and  $\alpha_5 \neq 0$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \exp \left[ \alpha_4 (\xi + \xi_0) \right] - \frac{\alpha_5}{\alpha_4}, \quad (55)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.5):** When  $\alpha_3 = \alpha_4 = 0$  and  $\alpha_5 \neq 0$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \alpha_5 (\xi + \xi_0), \quad (56)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.6):** When  $\alpha_3 = \alpha_5 = 0$  and  $\alpha_4 \neq 0$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \exp \left[ \alpha_4 (\xi + \xi_0) \right], \quad (57)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.7):** When  $\alpha_4 = \alpha_5 = 0$  and  $\alpha_3 \neq 0$ , the general solution of the equation (48) is:

$$\Omega(\xi) = -\frac{1}{\alpha_3 (\xi + \xi_0)}, \quad (58)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Case (1.8):** When  $\alpha_3 = \alpha_4 = \alpha_5 = 0$ , the general solution of the equation (48) is:

$$\Omega(\xi) = \xi_0, \quad (59)$$

where  $\xi_0$  is an arbitrary constant of integration.

**Remark (2):** The three cases (1.6), (1.7) and (1.8) are the spacial cases from the three cases (1.4), (1.1.3) and (1.5), respectively.

Every solution from above give the solution of Einstein field equation. Therefore, we will study some of these solutions as the following:

**Solution (1.1.1):** We consider the solution correspondence to the case (1.1.1). Without loss of generality, we can take, in this case,  $d = -c$ . Now, by using (49), (47), (44) and (40), we can find the solution as the following:

$$\begin{cases} A(x, t) = q_1 \exp \left[ c_7 \left( x + n_1(x + t) \right) \right] \cos^{2n_1} \left[ K_1 (x + t) + \xi_0 \right], \\ B(x, t) = q_2 \exp \left[ \frac{c}{2} \left( x - t + n_2(x + t) \right) \right] \cos^{n_1} \left[ K_1 (x + t) + \xi_0 \right], \\ C(x, t) = q_3 \exp \left[ \frac{c}{2} \left( t - x + n_2(x + t) \right) \right] \cos^{n_1} \left[ K_1 (x + t) + \xi_0 \right], \end{cases} \quad (60)$$

where  $n = \frac{c^2 + c_7^2 + 4 K_1^2}{4 c^2}$ ,  $n_1 = -\frac{c^2}{c_7^2 + 4 K_1^2}$  and  $n_2 = -\frac{c c_7}{c_7^2 + 4 K_1^2}$ . It is observed from equations (60), the line element (1) can be written in the following form:

$$ds_{111}^2 = q_1^2 \exp \left[ 2 c_7 (x + n n_1 (x + t)) \right] \cos^{4 n n_1} [\xi] (dx^2 - dt^2) \\ + \exp \left[ c (x - t + n_2 (x + t)) \right] \cos^{2 n_1} [\xi] \left( q_2^2 dy^2 + q_3^2 \exp \left[ 2 c (t - x) \right] dz^2 \right). \quad (61)$$

where  $\xi = K_1 (x + t) + \xi_0$  while  $q_1, q_2, q_3, c, c_7, K_1$  and  $\xi_0$  are an arbitrary constants.

**Solution (1.1.2):** We consider the solution correspondence the case (1.1.2). Without loss of generality, we can take, in this case,  $d = -c$ . Now, by using (50), (47), (44) and (40), we can find the solution as the following:

$$\begin{cases} A(x, t) = q_1 \exp \left[ c_7 (x + n n_1 (x + t)) \right] \cosh^{2 n n_1} [K_2 (x + t) + \xi_0], \\ B(x, t) = q_2 \exp \left[ \frac{c}{2} (x - t + n_2 (x + t)) \right] \cosh^{n_1} [K_2 (x + t) + \xi_0], \\ C(x, t) = q_3 \exp \left[ \frac{c}{2} (t - x + n_2 (x + t)) \right] \cosh^{n_1} [K_2 (x + t) + \xi_0], \end{cases} \quad (62)$$

where  $n = \frac{c^2 + c_7^2 - 4 K_2^2}{4 c^2}$ ,  $n_1 = -\frac{c^2}{c_7^2 - 4 K_2^2}$  and  $n_2 = -\frac{c c_7}{c_7^2 - 4 K_2^2}$ . It is observed from equations (62), the line element (1) can be written in the following form:

$$ds_{112}^2 = q_1^2 \exp \left[ 2 c_7 (x + n n_1 (x + t)) \right] \cosh^{4 n n_1} [\xi] (dx^2 - dt^2) \\ + \exp \left[ c (x - t + n_2 (x + t)) \right] \cosh^{2 n_1} [\xi] \left( q_2^2 dy^2 + q_3^2 \exp \left[ 2 c (t - x) \right] dz^2 \right). \quad (63)$$

where  $\xi = K_2 (x + t) + \xi_0$  while  $q_1, q_2, q_3, c, c_7, K_1$  and  $\xi_0$  are an arbitrary constants.

**Solution (1.1.3):** We consider the solution correspondence the case (1.1.3). Without loss of generality, we can take, in this case,  $d = -c$  and  $a = 1$ . Now, by using (51), (47), (44) and (40), we can find the solution as the following:

$$\begin{cases} A(x, t) = q_1 (x + t)^{-2 n d_2^2} \exp \left[ 2 d_1 d_2 (n t + n_1 x) \right], \\ B(x, t) = q_2 (x + t)^{-d_2^2} \exp \left[ n_+ t + n_- x \right], \\ C(x, t) = q_1 (x + t)^{-d_2^2} \exp \left[ n_- t + n_+ x \right], \end{cases} \quad (64)$$

where  $n = \frac{d_2^2 + 1}{4 d_2^2}$ ,  $n_1 = \frac{d_2^2 - 3}{4 d_2^2}$  and  $n_{\pm} = d_1 (d_2 \pm 1)$ . It is observed from equations (64), the line element (1) can be written in the following form:

$$ds_{113}^2 = q_1^2 (x + t)^{-4 n d_2^2} \exp \left[ 4 d_1 d_2 (n t + n_1 x) \right] (dx^2 - dt^2) \\ + (x + t)^{-2 d_2^2} \left( q_2^2 \exp \left[ 2 (n_+ t + n_- x) \right] dy^2 + q_3^2 \exp \left[ 2 (n_- t + n_+ x) \right] dz^2 \right), \quad (65)$$

where  $q_1, q_2, q_3, d_1$  and  $d_2$  are an arbitrary constants.

**Solution (1.2):** We consider the solution correspondence the case (1.2). Without loss of generality, we can take, in this case,  $d = -c$ ,  $c_7 = 2 n c$ ,  $n = \frac{1}{5}$  and  $a = -5 a_1$ . Now, by using (52), (47), (44) and (40), we can find



the solution as the following:

$$\begin{cases} A(x, t) = q_1 \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^2 e^{-\frac{c}{5}(5t+7x)}, \\ B(x, t) = q_2 \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^5 e^{-c(3t+2x)}, \\ C(x, t) = q_3 \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^5 e^{-c(2t+3x)}. \end{cases} \quad (66)$$

It is observed from equations (66), the line element (1) can be written in the following form:

$$\begin{aligned} ds_{113}^2 = & q_1^2 \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^4 e^{-\frac{2c}{5}(5t+7x)} (dx^2 - dt^2) \\ & + \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{10} \left( q_2^2 e^{-2c(3t+2x)} dy^2 + q_3^2 e^{-2c(2t+3x)} dz^2 \right), \end{aligned} \quad (67)$$

where  $q_1, q_2, q_3, c$  and  $a_1$  are an arbitrary constants.

**Solution (1.4):** We consider the solution correspondence the case (1.4). Without loss of generality, we can take, in this case,  $d = -c$ ,  $c_7 = b_2 a$ ,  $c = 2b_1$  and  $n = \frac{1}{4}$ . Now, by using (55), (47), (44) and (40), we can find the solution as the following:

$$\begin{cases} A(x, t) = q_1 \exp \left[ \frac{1}{2b_2} \left( 2b_2^2 x - b_1^2 (x+t) + e^{b_2(x+t)} \right) \right], \\ B(x, t) = q_2 \exp \left[ \frac{1}{b_2} \left( b_1 [2b_2 x - (b_1 + b_2)(x+t)] + e^{b_2(x+t)} \right) \right], \\ C(x, t) = q_3 \exp \left[ \frac{1}{b_2} \left( b_1 [b_2(t-x) - b_1(x+t)] + e^{b_2(x+t)} \right) \right]. \end{cases} \quad (68)$$

It is observed from equations (68), the line element (1) can be written in the following form:

$$ds_{14}^2 = A^2 (dx^2 - dt^2) + B^2 dy^2 + C^2 dz^2, \quad (69)$$

where  $q_1, q_2, q_3, b_1$  and  $b_2$  are an arbitrary constants.

**Solution (1.5):** We consider the solution correspondence the case (1.5). Without loss of generality, we can take, in this case,  $d = -c$ ,  $c_7 = 0$ ,  $c = 2\sqrt{2}a_3$ , and  $n = \frac{1}{4}$ . Now, by using (55), (47), (44) and (40), we can find the solution as the following:

$$\begin{cases} A(x, t) = q_1 \exp \left[ \left( \frac{x+t}{2} \right) \left( a_2 + \sqrt{2}a_3 + a_3^2(x+t) \right) \right], \\ B(x, t) = q_2 \exp \left[ a_2(x+t) + a_3^2(x+t)^2 + 2\sqrt{2}a_3 x \right], \\ C(x, t) = q_3 \exp \left[ a_2(x+t) + a_3^2(x+t)^2 + 2\sqrt{2}a_3 t \right]. \end{cases} \quad (70)$$

It is observed from equations (70), the line element (1) can be written in the following form:

$$\begin{aligned} ds_{15}^2 = & q_1^2 \exp \left[ (x+t) \left( a_2 + \sqrt{2}a_3 + a_3^2(x+t) \right) \right] (dx^2 - dt^2) \\ & + \exp \left[ 2(x+t) \left( a_2 + a_3^2(x+t) \right) \right] \left( q_2^2 e^{4\sqrt{2}a_3 x} dy^2 + q_3^2 e^{4\sqrt{2}a_3 t} dz^2 \right), \end{aligned} \quad (71)$$

where  $q_1, q_2, q_3, a_2$  and  $a_3$  are an arbitrary constants.

**Case (2):** When  $c_1 \neq 0$ , the similarity variable and similarity functions can be written as the following:

$$\xi = \frac{x+a}{t+b}, \quad B(x, t) = (x+a)^c \Psi(\xi), \quad C(x, t) = (x+a)^d \Phi(\xi), \quad (72)$$

where  $a = \frac{c_2}{c_1}$ ,  $b = \frac{c_3}{c_1}$ ,  $c = \frac{c_4}{c_1}$  and  $d = \frac{c_5}{c_1}$  are an arbitrary constants. Substituting the transformations (72) in the field Eqs. (20)-(21) lead to the following system of ordinary differential equations:

$$\begin{aligned} \frac{\xi \Psi'' + [1+c-c_9-2n(c+d)] \Psi'}{\xi \Psi} + \frac{\xi \Phi'' + [1+c-c_9-2n(c+d)] \Phi'}{\xi \Phi} \\ = 2n \left( \frac{\Psi'}{\Psi} + \frac{\Phi'}{\Phi} \right)^2, \end{aligned} \quad (73)$$

$$\begin{aligned} \frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi} + \frac{2[(2n+1)\xi^2 - c] \Psi'}{(2n+1)(\xi^2 - 1)\xi \Psi} + \frac{2[(2n+1)\xi^2 - d] \Phi'}{(2n+1)(\xi^2 - 1)\xi \Phi} \\ - \frac{2n}{2n+1} \left( \frac{\Psi'^2}{\Psi^2} + \frac{\Phi'^2}{\Phi^2} \right) = \frac{c(c-1) + d(d-1) - 2c_9 - 2n(c+d)}{(2n+1)(\xi^2 - 1)\xi^2}, \end{aligned} \quad (74)$$

where  $f(x) = k(x+a)^{c_9}$ ,  $k = c_8 c_1^{c_9}$ .

If one solves the system of second order non-linear ordinary differential equations (73)-(74), he can obtain the exact solutions of the original Einstein field equations (20)-(21) corresponding to reduction (72). The system (73)-(74) is very difficult to solve in general form. This system may be solved in some special cases in the future work.

## 5 Physical and geometrical properties of some models

**For the Model (61):**

The expressions for energy density  $\rho$ , the string tension density  $\lambda$ , magnetic permeability  $\bar{\mu}$  and The particle density  $\rho_p$ , for the model (61), are given by:

$$\rho(x, t) = -\lambda(x, t) = n_1 \exp \left[ 2c_7 \left( n(x+t) - x \right) \right] \sec^{n_1} [\xi] \left( n_3 \cos [\xi] - n_4 \sin [\xi] \right), \quad (75)$$

$$\bar{\mu}(x, t) = \frac{\chi F_{12}^2(x) \exp \left[ c \left( t - x - n_1(x+t) \right) \right]}{2c n_1 q_2^2 \cos^{2n_1} [\xi] \left( 2K_1 \tan [\xi] - c_7 \right)}, \quad (76)$$

$$\rho_p(x, t) = n_5 \exp \left[ 2c_7 \left( n(x+t) - x \right) \right] \cos^n [\xi], \quad (77)$$

where  $n_3 = \frac{c_7(c+c_7)+4K_1^2}{\chi q_1^2}$ ,  $n_4 = \frac{2cK_1}{\chi q_1^2}$ ,  $n_5 = \frac{2c^2}{q_1^2}$ ,  $\xi = K_1(x+t) + \xi_0$  and  $F_{12}(x)$  is an arbitrary function of the variable  $x$ .

The volume element is

$$V = q_1^2 q_2 q_3 \exp \left[ \frac{c_7}{2} \left( 3x - t + 3n_1(x+t) \right) \right] \cos^{3n_1-1} [\xi]. \quad (78)$$

The expansion scalar, which determines the volume behavior of the fluid, is given by:

$$\Theta = \left( \frac{5n_1-1}{4q_1} \right) \left( c_7 \cos [\xi] - 2K_1 \sin [\xi] \right) \exp \left[ c_7 \left( n(x+t) - x \right) \right] \sec^{n_1-1} [\xi], \quad (79)$$

The non-vanishing components of the shear tensor,  $\sigma_i^j$ , are:

$$\sigma_1^1 = \frac{2(n_1 + 1)\Theta}{3(1 - 5n_1)}. \quad (80)$$

$$\sigma_2^2 = \frac{\Theta}{3(5n_1 - 1)} \left[ n_1 + 1 - \frac{6c \cos[\xi]}{c_7 \cos[\xi] - 2K_1 \sin[\xi]} \right], \quad (81)$$

$$\sigma_3^3 = -(\sigma_1^1 + \sigma_2^2). \quad (82)$$

The shear scalar is:

$$\sigma^2 = \frac{\left[ n_6 + n_7 \cos[\xi] - 4c_7 K_1 \sin[\xi] \right] \Theta^2}{6(1 - 5n_1)^2 \left( c_7 \cos[\xi] - 2K_1 \sin[\xi] \right)^2}, \quad (83)$$

where  $n_6 = (n_1^2 - 10n_1 + 1)(c_7^2 + 4K_1^2)$  and  $n_7 = (n_1^2 + 1)(c_7^2 - 4K_1^2) - 2n_1(5c_7^2 + 28K_1^2)$ .

The acceleration vector is given by:

$$\dot{u}_i = \frac{1}{4} \left( c_7(n_1 + 3) + 2K_1(1 - n_1) \tan[\xi], 0, 0, 0 \right). \quad (84)$$

The deceleration parameter is given by [18, 19]

$$\begin{aligned} \mathbf{q} &= -3\Theta^2 \left( \Theta_{;i} u^i + \frac{1}{3} \Theta^2 \right) \\ &= \frac{(5n_1 - 1)^3}{256q_1^4} \exp \left[ 4c_7(n(x+t) - x) \right] \left( c_7 \cos[\xi] - 2K_1 \sin[\xi] \right) \\ &\quad \times \left( c^2 - c_7^2 + 20K_1^2 - (n_1 + 1) \left[ (c_7^2 - 4K_1^2) \cos[\xi] - 4c_7 K_1 \sin[\xi] \right] \right). \end{aligned} \quad (85)$$

**Remark (3):** It is worth noting that: If we put the following transformation

$$K_1 \rightarrow \imath K_2, \quad \sin \rightarrow \sinh, \quad \cos \rightarrow \cosh,$$

in the model (61), we have obtained the model (63), where  $\imath = \sqrt{-1}$ . Therefore, we can find the physical properties of the model correspondence to case (1.1.2) by putting the above transformation in the the model correspondence to case (1.1.1).

#### For the Model (65):

The expressions for energy density  $\rho$ , the string tension density  $\lambda$ , magnetic permeability  $\bar{\mu}$  and The particle density  $\rho_p$ , for the model (65), are given by:

$$\rho(x, t) = -\lambda(x, t) = \frac{4d_1}{\chi q_1^2} (x+t)^{d_2^2} \left[ d_2^2 - n_+(x+t) \right] \exp \left[ -4d_1 d_2 (nt + n_1 x) \right], \quad (86)$$

$$\bar{\mu}(x, t) = \frac{\chi F_{12}^2(x) (x+t)^{1+2d_2^2} \exp \left[ -4(n_+ t + n_- x) \right]}{8d_1 d_2 q_2^2 \left[ d_1(x+t) - d_2 \right]}, \quad (87)$$

$$\rho_p(x, t) = \frac{8d_1^2}{q_1^2} (x+t)^{1+d_2^2} \exp \left[ -4d_1 d_2 (nt + n_1 x) \right], \quad (88)$$

where  $F_{12}(x)$  is an arbitrary function of the variable  $x$ .

The volume element is

$$V = q_1^2 q_2 q_3 (x+t)^{-1-3d_2^2} \exp \left[ d_1 \left( (1+3d_2^2)t + 3(d_2^2 - 1)x \right) \right]. \quad (89)$$

The expansion scalar, which determines the volume behavior of the fluid, is given by:

$$\Theta = \frac{(5d_2^2 + 1) [d_1(x+t) - d_2]}{2d_2q_1} (x+t) \frac{d_2^2 - 1}{2} \exp \left[ -2d_1d_2(nt + n_1x) \right], \quad (90)$$

The non-vanishing components of the shear tensor,  $\sigma_i^j$ , are:

$$\sigma_1^1 = \frac{2(1 - d_2^2)\Theta}{3(1 - 5d_2^2)}. \quad (91)$$

$$\sigma_2^2 = \frac{[d_2(1 + d_2) + d_1(1 - 6d_2 - d_2^2)(x+t)]\Theta}{3(1 + 5d_2^2)[d_2 - d_1(x+t)]}, \quad (92)$$

$$\sigma_3^3 = -(\sigma_1^1 + \sigma_2^2). \quad (93)$$

The shear scalar is:

$$\sigma^2 = \frac{\Theta^2}{3(1 + 5d_2^2)^2} \left[ 1 + d_2^2(10 + d_2^2) + \frac{12d_2^3(2d_1(x+t) - 2d_2 + 1)}{(d_1(x+t) - d_2)^2} \right]. \quad (94)$$

The acceleration vector is given by:

$$\dot{u}_i = 2 \left( n_1 d_1 - \frac{n d_2}{x+t}, 0, 0, 0 \right). \quad (95)$$

The deceleration parameter is given by:

$$\begin{aligned} \mathbf{q} = & -\frac{(5d_2^2 + 1)^3}{8d_2^4q_1^4} (x+t)^{2(d_2^2-1)} \exp \left[ -8d_1d_2(nt + n_1x) \right] \\ & \times \left( d_2^2(d_2^2 + 2) + d_1(d_2^2 - 1)(x+t) [d_1(x+t) - 2d_2] \right). \end{aligned} \quad (96)$$

#### For the Model (67):

The expressions for energy density  $\rho$ , the string tension density  $\lambda$ , magnetic permeability  $\bar{\mu}$  and the particle density  $\rho_p$ , for the model (67), are given by:

$$\rho(x, t) = -\lambda(x, t) = \frac{6a_1c^2e^{\frac{2c(5t+7x)}{5}}}{\chi q_1} \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{-5}, \quad (97)$$

$$\bar{\mu}(x, t) = \frac{\chi F_{12}^2(x) e^{2c(3t+2x)}}{2c^2q_2^2} \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{-9} \left[ 5a_1 - e^{\frac{3c(x+t)}{5}} \right]^{-1}, \quad (98)$$

$$\rho_p(x, t) = \frac{2ce^{\frac{2c(5t+7x)}{5}}}{q_1^2} \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{-4}, \quad (99)$$

where  $F_{12}(x)$  is an arbitrary function of the variable  $x$ .

The volume element is

$$V = q_1^2 q_2 q_3 e^{\frac{-c(35t+39x)}{5}} \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{14}. \quad (100)$$

The expansion scalar, which determines the volume behavior of the fluid, is given by:

$$\Theta = -\frac{6c}{5q_1} e^{\frac{c(5t+7x)}{5}} \left[ 5a_1 + e^{\frac{3c(x+t)}{5}} \right] \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{-3}, \quad (101)$$

The non-vanishing components of the shear tensor,  $\sigma_i^j$ , are:

$$\sigma_1^1 = \frac{\Theta}{6}, \quad (102)$$

$$\sigma_2^2 = \frac{\Theta}{6} \left[ 5a_1 - 2e^{\frac{3c(x+t)}{5}} \right] \left[ 5a_1 + e^{\frac{3c(x+t)}{5}} \right]^{-1}, \quad (103)$$

$$\sigma_3^3 = -(\sigma_1^1 + \sigma_2^2). \quad (104)$$

The shear scalar is:

$$\sigma^2 = \frac{\Theta^2}{36} \left[ 25a_1^2 - 5a_1 e^{\frac{3c(x+t)}{5}} + 7e^{\frac{6c(x+t)}{5}} \right] \left[ 5a_1 + e^{\frac{3c(x+t)}{5}} \right]^{-2}. \quad (105)$$

The acceleration vector is given by:

$$\dot{u}_i = -\frac{c}{7} \left[ 7a_1 - e^{\frac{3c(x+t)}{5}} \right] \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{-1} (1, 0, 0, 0). \quad (106)$$

The deceleration parameter is given by:

$$\begin{aligned} \mathbf{q} = & -\frac{18c^2\Theta^2}{25q_1^2} e^{\frac{2c(5t+7t)}{5}} \left[ a_1 - e^{\frac{3c(x+t)}{5}} \right]^{-6} \\ & \times \left[ 25a_1^2 - 8a_1 e^{\frac{3c(x+t)}{5}} + e^{\frac{6c(x+t)}{5}} \right]. \end{aligned} \quad (107)$$

#### For the Model (69):

The expressions for energy density  $\rho$ , the string tension density  $\lambda$ , magnetic permeability  $\bar{\mu}$  and the particle density  $\rho_p$ , for the model (69), are given by:

$$\begin{aligned} \rho(x, t) = -\lambda(x, t) = & -\frac{4b_1}{\chi b_2 q_1^2} \left[ b_1 (b_1 + b_2) - b_2 e^{b_2(x+t)} \right] \\ & \times \exp \left[ \frac{1}{2b_2} \left( b_1^2 (x+t) - 2b_2^2 - e^{b_2(x+t)} \right) \right], \end{aligned} \quad (108)$$

$$\bar{\mu}(x, t) = \frac{\chi b_2 F_{12}^2(x)}{8b_1 q_2^2 \left[ b_1 - b_2 e^{b_2(x+t)} \right]} \exp \left[ \frac{2}{b_2} \left[ b_1 \left( b_2 (t-x) + b_1 (x+t) \right) - e^{b_2(x+t)} \right] \right], \quad (109)$$

$$\rho_p(x, t) = \frac{8b_1^2}{q_1^2} \exp \left[ \frac{1}{b_2} \left( b_1^2 (x-t) - 2b_2^2 x - e^{b_2(x+t)} \right) \right], \quad (110)$$

where  $F_{12}(x)$  is an arbitrary function of the variable  $x$ .

The volume element is

$$V = q_1^2 q_2 q_3 \exp \left[ \frac{1}{b_2} \left( 2b_2^2 x - 3b_1^2 (x-t) + 3e^{b_2(x+t)} \right) \right]. \quad (111)$$

The expansion scalar, which determines the volume behavior of the fluid, is given by:

$$\Theta = \frac{5}{2b_2 q_1} \left[ b_2 e^{b_2(x+t)} - b_1^2 \right] \exp \left[ \frac{1}{2b_2} \left( b_1^2 (x+t) - 2b_2^2 x - e^{b_2(x+t)} \right) \right], \quad (112)$$

The non-vanishing components of the shear tensor,  $\sigma_i^j$ , are:

$$\sigma_1^1 = -\frac{2\Theta}{15}, \quad (113)$$

$$\sigma_2^2 = \frac{\Theta}{15} \left( \frac{b_1^2 + 6b_1 b_2 - b_2 e^{b_2(x+t)}}{b_1^2 - b_2 e^{b_2(x+t)}} \right), \quad (114)$$

$$\sigma_3^3 = -(\sigma_1^1 + \sigma_2^2). \quad (115)$$

The shear scalar is:

$$\sigma^2 = \frac{\Theta^2}{25} \left[ \frac{1}{3} + 4b_1^2 b_2^2 \left( b_1^2 - b_2 e^{b_2(x+t)} \right)^{-2} \right]. \quad (116)$$

The acceleration vector is given by:

$$\dot{u}_i = \frac{1}{2d_1} \left[ 2b_2^2 - b_1^2 + b_2 e^{b_2(x+t)} \right] (1, 0, 0, 0). \quad (117)$$

The deceleration parameter is given by:

$$\begin{aligned} \mathbf{q} = & -\frac{5\Theta^2}{2b_2^2 q_1^2} \left[ b_1^2 + b_2 (3b_2^2 - 2b_1^2) e^{b_2^2(x+t)} + 2b_2^2 e^{b_2(x+t)} \right] \\ & \times \exp \left[ -\frac{1}{b_2} \left( 2b_2^2 - b_1^2(x+t) + e^{b_2(x+t)} \right) \right]. \end{aligned} \quad (118)$$

#### For the Model (71):

The expressions for energy density  $\rho$ , the string tension density  $\lambda$ , magnetic permeability  $\bar{\mu}$  and the particle density  $\rho_p$ , for the model (71), are given by:

$$\rho(x, t) = -\lambda(x, t) = \frac{4\sqrt{2}a_3}{\chi q_1^2} \left[ a_2 + 2a_3^2(x+t) \right] \exp \left[ -(x+t) \left( a_2 + \sqrt{2}a_3 + a_3^2(x+t) \right) \right], \quad (119)$$

$$\bar{\mu}(x, t) = -\frac{\chi F_{12}^2(x)}{8\sqrt{2}a_3 q_2^2} \left[ \sqrt{2}a_3 + a_2 + 2a_2^2(x+t) \right]^{-1} \quad (120)$$

$$\times \exp \left[ -2 \left( 2\sqrt{2}a_3 + a_2(x+t) + a_3^2(x+t)^2 \right) \right],$$

$$\rho_p(x, t) = \frac{16a_3^2}{q_1^2} \exp \left[ -(x+t) \left( \sqrt{2}a_3 + a_2 + a_3^2(x+t) \right) \right], \quad (121)$$

where  $F_{12}(x)$  is an arbitrary function of the variable  $x$ .

The volume element is

$$V = q_1^2 q_2 q_3 \exp \left[ 3(x+t) \left( \sqrt{2}a_3 + a_2 + a_3^2(x+t) \right) \right]. \quad (122)$$

The expansion scalar, which determines the volume behavior of the fluid, is given by:

$$\Theta = \frac{5}{2q_1} \left[ \sqrt{2}a_3 + a_2 + 2a_3^2(x+t) \right] \exp \left[ -\left( \frac{x+t}{2} \right) \left( \sqrt{2}a_3 + a_2 + a_3^2(x+t) \right) \right], \quad (123)$$

The non-vanishing components of the shear tensor,  $\sigma_i^j$ , are:

$$\sigma_1^1 = -\frac{2\Theta}{15}, \quad (124)$$

$$\sigma_2^2 = \frac{\Theta}{15} \left( \frac{a_2 - 5\sqrt{2}a_3 + 2a_3^2(x+t)}{a_2 + \sqrt{2}a_3 + 2a_3^2(x+t)} \right), \quad (125)$$

$$\sigma_3^3 = -(\sigma_1^1 + \sigma_2^2). \quad (126)$$

The shear scalar is:

$$\sigma^2 = \frac{\Theta^2}{25} \left[ \frac{1}{3} + 8a_3^2 \left( a_2 + \sqrt{2}a_3 + 2a_3^2(x+t) \right)^{-2} \right]. \quad (127)$$

The acceleration vector is given by:

$$\dot{u}_i = \left[ \frac{a_3}{\sqrt{2}} + \frac{a_2}{2} + a_3^2(x+t) \right] (1, 0, 0, 0). \quad (128)$$

The deceleration parameter is given by:

$$\mathbf{q} = -\frac{5\Theta^2}{2q_1^2} \left[ 2\sqrt{2}a_3a_2 + a_2^2 + 8a_3^2 + 4a_3^2(\sqrt{2}a_3 + a_2)(x+t) + 4a_3^4(x+t)^2 \right] \\ \times \exp \left[ -(x+t) \left( a_2 + \sqrt{2}a_3 + a_3^2(x+t) \right) \right]. \quad (129)$$

## 6 Conclusion

In the paper, we have derived some new invariant solutions of Einstein-Maxwell's field equations for string fluid as source of matter in cylindrically symmetric space-time with Variable Magnetic Permeability. Different set of solutions are found using different values of the parameters. Note that the cosmological solutions are physically viable for the following reasons:

- [i] energy density is positive and decreasing with the increase of the time
- [ii] volume of the Universe is increasing due the expanding nature of the Universe
- [iii] deceleration parameter should be negative as recent observations indicate that our Universe is accelerating
- [iv]  $\frac{\sigma}{\theta}$  will be vanished at large time as the Universe may got isotropized in some later time
- [v] solutions must be non singular as existence of Big-bang singularity is one of the basic failures of general theory of relativity

The models (61), (65), (67) and (69) do not meet the above criterion (i). Here, in the models (61) and (65), densities increase with time whereas in models (67) particle density is increasing and in model (69) particle density is negative (Fig-1). Therefore, these models are not physically interesting. On the other hand, the model (71) is very much acceptable as it describes more or less observable Universe. It is to be noted that our procedure of solving the field equations (symmetry analysis method) is completely different compare to usual methods available in literature. The derived model starts expanding without Big Bang singularity ( fig-2). Also, from the theoretical perspective, the present model can be a viable model to explain the acceleration of the Universe. In other words, the solution presented here can be one of the potential candidates to describe the observed Universe. In our model, it seems magnetic field with negative magnetic Permeability (Fig-3) is responsible to provide accelerated as well as singularity free Universe. Nowadays, negative magnetic permeability is not an impossible event rather it can be found in split ring resonator (SRR) in the visible light region [20]. We also note that  $\frac{\sigma}{\theta}$  will be vanished at large time (Fig-2). This means the Universe may got isotropized in some later time. Again, one can assume that negative magnetic Permeability is responsible for the isotropisation. A detail discussions of all basic cosmological constraints is beyond the scope in the present paper. We hope it will be taken care in a future project.

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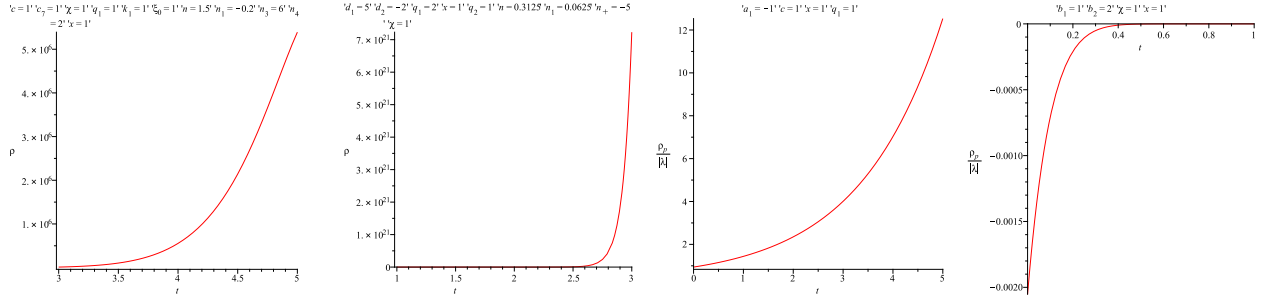


Figure 1: (Left) The variation of energy density with respect to time for the model(61). (First Middle) The variation of energy density with respect to time for the model(65). (Second Middle) The variation of particle density with respect to time for model (67). (Right) The variation of particle density with respect to time for model (69).

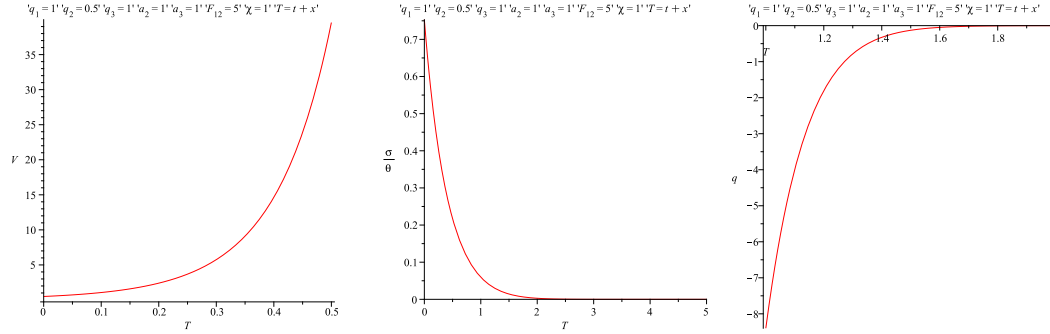


Figure 2: (Left) The variation of the volume of the Universe with respect to time.. (Middle) The variation of  $\frac{\sigma}{\theta}$  with respect to time. (Right) The deceleration parameter is shown with respect to time.

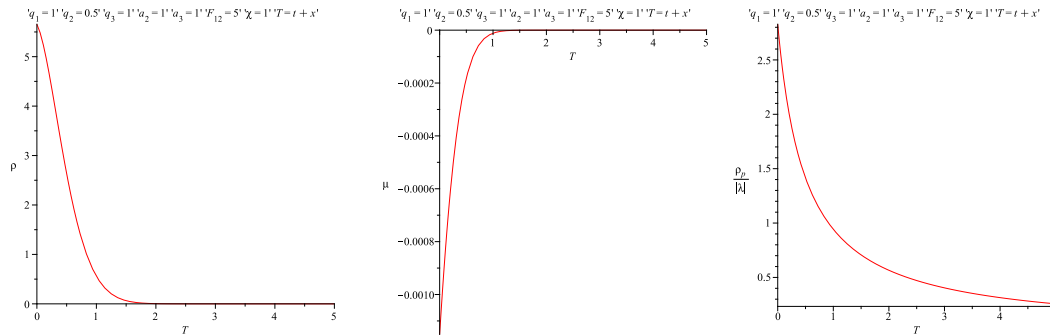


Figure 3: (Left) The variation of energy density with respect to time. (Middle) The variation of magnetic Permeability with respect to time. (Right) The variation of particle density with respect to time.



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